THE GROUPS OF STUDENTS PROBLEM: INSIGHTS ABOUT MULTIPLICATION AND IMPLIED ORDER IN COMBINATORIAL ENUMERATION

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Counting problems have applications in probability and computer science, and they provide rich contexts for problem solving. Such problems are accessible to students, but subtleties can arise that make them surprisingly difficult to solve. In this paper, students' work on the Groups of Students problem is presented, and an important issue related to multiplication (a fundamental aspect of counting) is discussed. Examples from two categories of students are presented – those who were able to make sense of a correct solution to the counting problem, and those who were not. Evidence is provided that many students failed to attend to the implied order in the multiplicative process, and a productive way of thinking that emerged for some students is also shared. Pedagogical implications and suggested avenues for further work are provided.

Keywords: Post-secondary Education, Advanced Mathematical Thinking, Probability

Introduction and Motivation

As one aspect of discrete mathematics, combinatorial enumeration (the study of counting problems) has received attention in both K-12 and undergraduate curricula, emerging as an important facet of problem solving and probability (e.g., CCSS, 2010; English, 2005). While counting problems are often simple to state and explore, they contain sophisticated mathematical ideas. In spite of their accessibility, there is evidence that students are often unsuccessful when solving counting problems (e.g., Eizenberg & Zaslavsky, 2004; Hadar & Hadass, 1981; Author, Date). Given that discrete and combinatorial topics are something with which mathematics and computer science students should have facility, there is a need for mathematics education researchers to look harder at students' work on counting problems and to attend to potential ways in which students may succeed on such tasks. In this paper, results from student work on one particular *Groups of Students* counting problem are reported. Through the exploration of student work on this problem, the research goal addressed in this paper is to present and discuss students' difficulty in recognizing a key issue related to multiplication – that multiplication implies a process with ordered stages. Some students never resolved this issue, but those who did used outcomes as a means of making sense of the issue. By focusing on one counting problem, we can go into significant mathematical detail, pulling apart some especially salient mathematical issues that might arise more broadly for students. The goal is to use the case of this particular counting problem to identify issues that might be more generalizable for students as they solve other counting problems. This work suggests that students' conceptual understanding of multiplication in counting may be limited, and that students' correct and incorrect assumptions about multiplication should be examined in greater detail.

Literature Review and Theoretical Perspective

There is ample evidence in the research literature that students struggle with correctly solving counting problems. For instance, Hadar & Hadass (1981) identify pitfalls that students face in solving counting problems, and Eizenberg and Zaslavsky (2004) report low success rates for undergraduates. Authors of combinatorics textbooks also underscore the difficulty of counting problems, as Martin (2001) calls the first chapter of his college-level textbook "Counting is Hard," and Tucker (2002) stresses that his counting chapter "is the most important and difficult chapter" in

his textbook on applied combinatorics. Several mathematics education researchers have tried to identify reasons for such struggles, examining strategies that might be productive for students (e.g., English, 1991), studying the effects of implicit combinatorial models (Batanero, Navarro-Pelayo, & Godino, 1997), and investigating students' verification of counting problems (Eizenberg & Zaslavsky, 2004). Halani (2012) looked beyond student difficulties in an attempt to identify ways of thinking that students employ in solving counting problems, and Author (Date) suggested that focusing on student-generated connections might be a productive way to understand how students conceptualize counting problems. In spite of such research, on the whole there is still much to learn about students' thinking related to counting, specific causes for student difficulties, and ways in which such difficulties might be allayed. In this paper, one potential factor of confusion related to multiplication in counting is highlighted.

Theoretically, this work is framed within a focus on mathematical issues of multiplication and order, and within Author's (Date) notion of a set-oriented perspective. Multiplication is undeniably foundational in counting problems. Informally, the multiplication principle is the notion that the total outcomes of a counting process is the product of the outcomes of independent stages of that process, and many refer to this as the "fundamental counting principle" (e.g., English, 2005; Sengadir, 2009). Tillema (2007) has focused on middle school students' multiplicative reasoning in the context of counting tasks, and his work suggests that multiplicative reasoning on such tasks is far from trivial. Students at a variety of levels often view multiplication as a simple task, and they tend to gloss over the details of why multiplication works on counting problems. However, while multiplication is often straightforward, situations can arise in which students encounter subtleties about multiplication in counting problems. In addition, issues of order are perennial in counting problems, and the question of whether or not outcomes should be ordered can determine which counting processes or formulas are appropriate. Several researchers have identified order as being a key element of students' counting (Batanero, et al., 1997; Mellinger, 2004), and the determination of whether or not order "matters" in a counting problem is not an insignificant task. Additionally, Author (Date) has argued for the importance of sets of outcomes as related to order. Author advocated a perspective that counting involves enumerating sets of outcomes, and that counting activity can be considered in light of structuring those outcomes. The findings below, and particularly the work of students who were successful at understanding the correct solution, are situated within this set-oriented perspective.

As the results and discussions show, the work described herein highlights a subtle aspect of multiplication – that multiplication can imply ordered stages in a counting process. This work emphasizes that this implied order in multiplication is perhaps not something to which students naturally attend. This paper presents student work on a task for which failure to attend to this important property of implied order in multiplication leads to problematic issues. The case is made that this aspect of students' multiplicative reasoning may deserve more explicit pedagogical attention and should be investigated further.

Methods

Participants. Twenty-two post-secondary students participated in individual, videotaped, 60-90 minute-long interviews. The students were junior, senior, and graduate mathematics students who had experience with fundamental combinatorial ideas such as binomial coefficients. The structure of these interviews was first to give students five counting problems to solve on their own. Then, students subsequently returned to these problems, during which time they were presented with alternative answers to evaluate (often these alternatives were reasonable but incorrect answers, developed in pilot work.) Typically, this resulted in students comparing two answers, both of which could seem reasonable, but one of which contained an error. This facilitated situations in which students evaluated incorrect but seemingly reasonable answers.

Tasks. The students each solved five problems, but in this paper the *Groups of Students* problem is emphasized. For this problem, both a correct answer and an incorrect answer are provided. While many correct and incorrect answers could be presented, two answers are given that are most relevant to subsequent discussion.

The Groups of Students Problem. The Groups of Students problem states, "In how many ways can you split a class of 20 into 4 groups of 5?" A correct answer to this problem is $\left[\binom{20}{4}\cdot\binom{15}{5}\cdot\binom{10}{5}\cdot\binom{5}{5}\right]/4!$. To arrive at this solution, five students are first chosen to be in a group, and there are $\binom{20}{5}$ ways to do this. Then, for each choice of the first five, five of the remaining students are chosen to be in another group, and there are $\binom{15}{5}$ ways to do this. Similarly, five more are chosen to be in a group, $\binom{10}{5}$, and then finally the last five to be in a group, $\binom{5}{5}$. These four terms are multiplied together because for each group chosen in a respective stage, there are a certain number of options available at each subsequent stage. However, it is noteworthy that this multiplication process assumes a first, second, third, and fourth group – the groups were developed in a particular order. Because of this, the product must divided by 4 factorial since the groups are not meant to be labeled or distinguished in any way. A typical incorrect answer is $\binom{20}{4}\cdot\binom{15}{5}\cdot\binom{10}{5}\cdot\binom{5}{5}$, which neglects the division by 4 factorial. Reasoning behind each of these answers are further detailed in the results section.

Data analysis. Initial data analysis involved transcription of the interviews. Then, the Groups of Students problem was targeted, and the author carefully analyzed and categorized students' solutions to this problem. In an initial pass different solutions and strategies were characterized, and the implied order in multiplication was identified as mathematical theme of interest. Videos excerpt that related to multiplication were re-examined, and a narrative (Auerbach & Silverstein, 2003) about the students' performances on the Groups of Students problem was constructed.

Results

The results are organized into two sections. First, overall numerical results are given, emphasizing that there was a low success rate on the problem. Second, to highlight particular mathematical issues that arose, work from two different students is presented, one who was ultimately *unable* to understand the correct answer, and one who was *able* to understand it.

Overall results. On the whole, students struggled with the Groups of Students problem. Only three of 22 students answered the question correctly on their first attempt (Table 1 provides a breakdown of students' initial answers). Discussions during the interviews confirmed that the students had interpreted the problem as intended (as involving splitting the groups into four indistinguishable (not distinguishable) groups of five students).

Table 1: Frequencies of Initial Answers

Expression	Initial Answer	Correct/Incorrect	Frequency (%)
Expression (1)	$\left[\binom{20}{4}\cdot\binom{15}{5}\cdot\binom{10}{5}\cdot\binom{5}{5}\right]/4!$	Correct	2 (9%)

Expression (2)	$\frac{20!}{5!\cdot 5!\cdot 5!\cdot 5!\cdot 4!}$	Correct	1 (5%)
Expression (3)	$\binom{20}{5}\cdot\binom{15}{5}\cdot\binom{10}{5}\cdot\binom{5}{5}$	Incorrect	8 (36%)
Expression (4)	$\binom{20}{5}\cdot\binom{15}{5}\cdot\binom{10}{5}$	Incorrect	3 (14%)
Expression (5)	$\binom{20}{5}+\binom{15}{5}+\binom{10}{5}+\binom{5}{5}$	Incorrect	2 (9%)
	Other	Incorrect	6 (27%)

Student data. In this subsection, work of one representative student is presented from each of two categories of students (see Table 2; student work from categories in bold are presented in detail). First, work is presented of one of seven students who initially answered the problem incorrectly and was *unable* to make sense of why division by 4 factorial was necessary. The student's inability to recognize the implicit order in multiplication is emphasized, and reasons for this phenomenon are suggested. Second, work is presented of one of nine students who initially answered the problem incorrectly but was *able* to make sense of division by 4 factorial. The student's utilization of a specific outcome allowed him to recognize the need for the division by 4 factorial. The two students' work is contrasted. In the Discussion section, major themes from these findings are synthesized and implications and avenues for further study are proposed.

Table 2: Categorization of Students

Category description	Frequency (%)
Students answered the problem correctly initially	3 (14%)
Students answered the problem incorrectly initially, were presented with a correct solution, and were UNABLE to understand the correct solution	**7 (31%)**
Students answered the problem incorrectly initially, were presented with a correct solution, and were ABLE to understand the correct solution	**9 (41%)**
Students answered the problem incorrectly initially; did not revisit the problem	3 (14%)

Nic – Initially incorrect, unable to make sense of division by 4 factorial. Nic originally arrived at the incorrect Expression 3. In his work he had written down Gp 1, Gp 2, Gp 3, Gp 4 (see his work in Figure 1**),** although he did not reference them in his answer. In fact, his discussion revealed that while he wrote different group numbers, he interpreted the problem correctly (that the groups were to be indistinguishable), saying that the question was asking about a situation, "where you're just splitting them into the groups and then that's it." Given his proper interpretation of the problem, the fact that he wrote down group numbers is noteworthy. He stated that he did so simply to "keep an order" in his head, but he did not seem to realize that this work suggested that he was unintentionally treating the groups as distinguishable.

> *Interviewer:* Okay. Cool, the Group 1, Group 2, Group 3, Group 4, did that help you in that counting process, or why did you distinguish between Group 1, Group 2, Group 3, Group 4?
> *Nic:* All it really did was to, basically I have 20 choose 5, so that gives me a group, is basically what I was thinking of, so I need to make sure I subtract those kids off for the next possible

group… So it was basically just helping me keep an order on where I was at in the process, so, it was more keeping an order for me in my head.

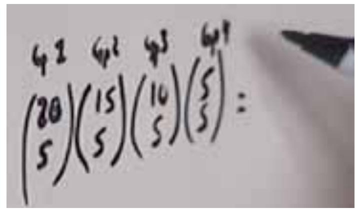

Figure 1: Nick's Group Labels Above Each Binomial Coefficient

When Nic revisited the problem and was given the correct Expression 1 to evaluate, he was unable to explain the division by 4 factorial. There was evidence that the unintended distinguishability of the groups seemed to be a key factor in his inability to reconcile the division by 4 factorial. In Nic's thinking, his process of writing down the labels for the groups was simply organizational, and he did not realize that by performing the multiplication, he was actually employing a counting process with implicitly ordered stages. The excerpt below provides evidence for why Nick might not have recognized that multiplication has implied ordered stages. He intimates that the binomial coefficient, the "choosing," already indicated that, "you're not permutating [sic]." This suggests that Nic felt the issue of order (and not making the groups distinguishable) was already handled in his answer to the problem.

Nic: In how many ways can you split a class of 20 into 4 groups of 5?...I think that [his initial answer, Expression 3] is still right. Because when you do this choosing, when you do this already takes into account that you're not permutating. This one [the alternative solution, Expression 1] you're doing it twice. So this [Expression 1] is probably undercounting I think.

There is further evidence that Nic intended for his labeling of the groups as helping him keep track of things, focus on groups and not students. However, he did not recognize that there was an implicit distinguishability in his answer through the multiplication. Additionally, he did not notice that by saying there was a Group 1, Group 2, Group 3, and Group 4, he was implying that there was a first, second, third, and fourth ordered stage to his process.

Interviewer: Okay, now in your solution you had written a Group 1, Group 2, Group 3, Group 4 above it. What was that?

Nic: Uh, that was just show that I was doing it by groups rather than students, that we have 4 different possibility – 4 different groups that the students could go into that wasn't anything for them to be – the groups weren't labeled or anything, that's just knowing that you have to split 20 things into 4 spots, rather than actual names.

Even after spending significant time on the problem, Nic was ultimately not able to understand or explain the correct Expression 3. Nic's work on the Groups of Students problem shows an instance of a student who initially answered the problem incorrectly, and who was unable to make sense of the division by 4 factorial, even when the correct answer was presented to him (he was one of seven such students). In seeing this struggle, what comes to light is the fact that there was confusion for Nic about whether his multiplication somehow already caused the groups to be distinguishable and ordered in some way. His inability to see this is addressed further in the Discussion section.

Owen – Initially incorrect, able to make sense of division by 4 factorial. Owen had initially generated Expression 4, and his explanation highlights the sequential order in which he came up with the answer. When asked why he did not write the final term of choosing 5 of 5 students, Owen said, "There's just one way to do that, there's only one. They're just sitting there in their group already."

Owen: Ah, interesting. 4 groups of 5. Well, this one I mean, it seems pretty easy to pick the first group, you know, I have 20 people, choose 5…Great. Well, this one I feel kind of friendly with,

‘cause then whoever I chose, I have 15 people left, I just choose 5…I only have 10 people left, and I choose 5. There’s 5 people left. That’s how I’d count it. I don’t know if that’s right or not.

When Owen returned to the problem later in the interview, he was given Expression 1 to evaluate. Upon reading it, he said, “I don’t understand why they would divide by 4 factorial I guess…I’m immediately confused by that.” After taking some time to think about what might have been going on, Owen was eventually able to explain what had happened in the problem, and why he would indeed need to divide by 4 factorial. To do so, Owen appealed to a particular outcome, showing how the incorrect Expression 4 could lead to an overcount. Owen started to write an example, but not wanting to write out 20 objects, he instead considered splitting up 8 students into four groups of two. He wrote down ABCDEFGH, and he wrote the corresponding solution that he would have done with 8 students, $\binom{8}{2}\cdot\binom{6}{2}\cdot\binom{4}{2}\cdot\binom{2}{2}$. He then took different colored pens, and, as he explained the solution below, he underlined each binomial coefficient with a different color (pink, red, orange, and green, respectively; see Figure 2).

Owen: I’m going to order my selections, because I think of them as happening in some order. So it’s like this first [draws a pink line under $\binom{8}{2}$], and I’m going to do this second [draws a red line under $\binom{6}{2}$], this third [draws an orange line under $\binom{4}{2}$], and this fourth [draws a green line under $\binom{2}{2}$], right

Figure 2: Owen’s Color Coding of the Outcome

Then, as seen in the excerpt below, Owen recognized that when he chose the groups of students to be AB, CD, EF, and GH, he could have chosen them in some order; the colors represented a particular order in which he chose that division of the 8 people. As he talked about choosing a pair of students, he drew a color under that pair. He drew lines under AB, then CD, then EF, then GH in that first example, and he drew lines under AB, then EF, then GH, then CD in the second.

Owen: What if, when I first chose 2 students, I picked, I’m going to make it really obvious, A and B [draws a pink line under AB]… Cool, then I have 6 remaining. Then I pick C and D [draws a red line under CD]. Oh, what if I have 4 remaining? Oh I’m going to pick these 2 [draws an orange line under EF, then draws a green line under GH].

He then noted that he could pick the same division of 8 people (AB, CD, EF, GH) in another way, and he drew colors above the letters to represent picking the pairs in a different order. This time, he drew AB first, then EF, then GH, then CD.

Owen: Well, that’s one way to count, pick ‘em, right? Color, change the colors drawn above again, so…you can pick the pairs in different orders… To make it obvious I’ll draw the colors. Look, I’m just drawing the same exact pairs, different colors up here…To symbolize that if I did

> this, I could have picked him, this same pair twice, right, but second I could have picked CD or I could have picked EF…And then third I could have picked EF or GH…And last, CD or GH, the remaining pairs…Same exact pairs, counted twice here [Expression 4], counted only once here [Expression 1] because you're dividing out the ordering of the pairs.

Owen is an example of a student who initially got the problem incorrect, but who was ultimately able to make sense of the correct Expression 1 when it was presented to him. Owen's work on this problem is noteworthy because it shows how he identified a particular outcome (or an outcome in a smaller case) in order to address the overcounting issue and ultimately decide which expression was correct. His work suggests that he recognized that the multiplication was imposing order on his process, producing distinguishable groups. He cleverly used a color-coding scheme to indicate exactly how the same division of 8 students was actually counted more than once by the incorrect expression. His work here suggested that focusing an example in the set of outcomes of the smaller problem allowed him to complete the problem correctly.

Discussion and Conclusion

The above findings show examples from two groups of students who faced varying levels of success on the Groups of Students problem. As they struggled with making sense of the division by 4 factorial in Expression 1, evidence emerged about an important aspect of students' thinking about multiplication. In particular, the Groups of Students problem revealed that students did not always recognize that multiplication necessarily implies ordered stages. The ordered stages are difficult to detect in this particular problem because the problem counts fairly abstract objects, and because the problem solution includes binomial coefficients. For instance, Nic assumed that his initial solution took into account the fact that he wanted the groups to be indistinguishable, and he assumed that the multiplication of the binomial coefficients did not imply ordered stages. This was presumably partly because of the presence of binomial coefficients (which tend to signify order *not* mattering), causing him not to realize that, whether he intended to or not, the act of multiplying was bringing order into the problem. This contributed to his failure to realize that he was really incorporating order, and, thus, to answer the problem correctly, he needed to account for the groups being distinguishable by dividing by 4 factorial.

The Groups of Students problem brings to light significant issues regarding multiplication and order, both of which are absolutely fundamental aspects of counting. Students' poor success on this problem (both initially, and even after being given the correct answer) suggests that multiplication is not as well understood as might be expected. Thus this problem highlights an important issue involving multiplication and suggests that this implied order in multiplication is something that should receive more explicit attention among students who are learning to count. The findings also suggest a way to fix this issue, however. Owen's work, for instance, suggested that while he initially could not understand the division by 4 factorial, it was an appeal to a particular outcome that enabled him to make sense of why the division was necessary. Indeed, there were 9 students who initially got the problem wrong but were ultimately able to justify the division by 4 factorial. For all 9 of them, they used particular outcomes (in a manner similar to Owen) to come to understand the correct solution. The role of outcomes in identifying an overcount contributes to a growing body of evidence (Author, Date) that outcomes provide a productive perspective for counters to emphasize. Thus, more can be done in helping students articulate how the multiplication counting process actually generates outcomes, and by tying multiplication to outcomes productive and correct conceptions of multiplication can be strengthened, particularly in the context of counting. In examining students who were unsuccessful, problematic ways of thinking emerged that could be suggestive of broader issues as students count, especially pertaining to a lack of understanding of implied ordered stages in multiplication. In examining students who were successful, a particularly productive way of thinking arose, as students drew upon the set of outcomes to effectively reconcile and explain a key

mathematical issue. These students were able to articulate how the multiplication and the order were interacting in the incorrect Expression 3 to generate some outcomes more than once.

Pedagogically, students may benefit from engaging with problems like the Groups of Students problem, which raise subtle aspects of important counting concepts like multiplication and order. Multiplication cannot merely being seen as a trivial, taken-for-granted process, but rather teachers must strive to develop deep multiplicative reasoning in students, explicitly tying properties of multiplication to counting processes.

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